Computational Imaging Course

Markov Random Fields, Inverse problems & Computational Imaging

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MCQ

1. **What is a Markov Random Field ?**
   - ❑ a particular case of Markov chain
   - ❑ a random process on undirected graphs
   - ❑ a Bayesian model
   - ❑ a random field whose distribution is written as a Gibbs distribution

2. **The (local) Markov property for random field states that the conditional distribution of a pixel given all other pixels:**
   - ❑ depends only on previous pixels
   - ❑ is a multivariate Gaussian distribution
   - ❑ depends only on neighboring pixels
   - ❑ is null

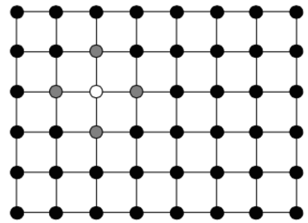
1. **What is a clique ?**
   - ❑ a particular case of Markov chain
   - ❑ a set of mutually-neighboring nodes on a graph
   - ❑ the set of all neighboring nodes
   - ❑ the set of all nodes of a graph

Main notions

- **Markov Random Fields**
- **Gibbs Random Fields**
- **Cliques & potential functions**
- **Bayesian models**

- **And also : Graph cut, ICM, Gibbs sampling**

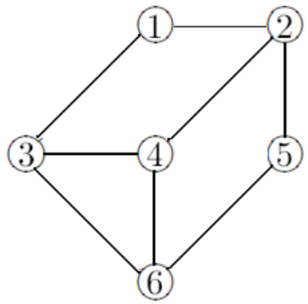
GraPH Definitions



- A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a finite collection of nodes (or vertices) $\mathcal{V} = \{n_1, n_2, \ldots, n_N\}$ and set of edges $\mathcal{E} \subset \binom{\mathcal{V}}{2}$
- We consider only undirected graphs
- Neighbor: Two nodes $n_i, n_j \in \mathcal{V}$ are neighbors if $(n_i, n_j) \in \mathcal{E}$
- Neighborhood of a node: $\mathcal{N}(n_i) = \{n_j : (n_i, n_j) \in \mathcal{E}\}$
- Neighborhood is a symmetric relation: $n_i \in \mathcal{N}(n_j) \Leftrightarrow n_j \in \mathcal{N}(n_i)$
- Complete graph: $\forall n_i \in \mathcal{V},\ \mathcal{N}(n_i) = \{(n_i, n_j), j = \{1, 2, \ldots, N\} \setminus \{i\}\}$
- Clique: a complete subgraph of $\mathcal{G}$.
- Maximal clique: Clique with maximal number of nodes; cannot add any other node while still retaining complete connectedness.

From Srinivas, 2011

GraPH Definitions



- $\mathcal{V} = \{1, 2, 3, 4, 5, 6\}$
- $\mathcal{E} = \{(1, 2), (1, 3), (2, 4), (2, 5), (3, 4), (3, 6), (4, 6), (5, 6)\}$
- $\mathcal{N}(4) = \{2, 3, 6\}$
- Examples of cliques: $\{(1), (3, 4, 6), (2, 5)\}$
- Set of all cliques: $\mathcal{V} \cup \mathcal{E} \cup \{3, 4, 6\}$

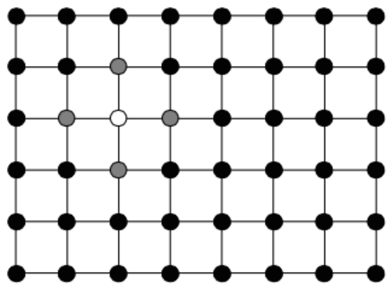
From Srinivas, 2011

Markov Random Fields

A Markov Random field is a set of random variables defined on a 2D lattice such that their joint probability distribution satisfies the local Markov property for any node of the 2D lattice:

$$p(x_i|\mathbf{x}_{\mathcal{V}\setminus\{i\}}) = p(x_i|\mathbf{x}_{\mathcal{N}(i)})$$

The grahical representation of a MRF is an undirected graphical model in which each node corresponds to a random variable or a collection of random variables, and the edges identify conditional dependencies.



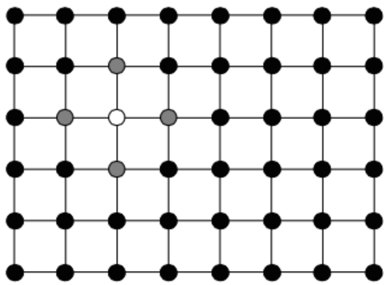
2D lattice with a 4-neighborhood structure

Gibbs Random Fields

A Gibbs Random field is a set of random variables defined on a 2D lattice such that their joint probability distribution :

$$p(\mathbf{x}) = \frac{1}{Z} \exp\left\{ -\sum_{C \in \mathcal{C}} V_C(\mathbf{x}_C) \right\},$$

with Z the partition function and V_C clique potentials



Ising model: binary MRF (+1:-1) on a 2D lattice

$$p(\mathbf{x}) = \frac{1}{Z} \exp\left\{ \beta \sum_{(i,j) \in \mathcal{E}} x_i x_j \right\}$$

Hammersley-clifford Theorem

Consider a random field x on a graph G, such that p(x) > 0. Let C denote the set of all maximal cliques of the graph.

If the field satisfies the local Markov property then p(x) can be writenn as a Gibbs distribution :

$$p(\mathbf{x}) = \frac{1}{Z} \exp\left\{ -\sum_{C \in \mathcal{C}} V_C(\mathbf{x}_C) \right\},$$

Conversely, if p(x) can be written as a Gibbs distribution, it verifies the local Markov property.

That is to say : Markov Random Fields ⇔ Gibbs Random Fields

sampling markov random fields

No direct sampling method as for instance for Gaussian random variables.

Principle: MCMC (Monte Carlo Markov Chain) approach to simulate a sequence of images such that this sequence converges in law to the target distribution p(x)

Gibbs sampler : transition matrix of the Markov chain designed such that at each iteration only one pixel value is modified.

$$p\left(X_i^{m+1} = x_i | X^m = x^m\right) = \delta(x_i - x_i^m),\ \forall i \neq i^*$$

$$p\left(X_{i^*}^{m+1} = x_{i^*} | X^m = x^m\right) = p\left(\mathbf{x}_{i^*} = x_{i^*} | \mathbf{x}_{\mathcal{N}(i^*)} = x_{\mathcal{N}(i^*)}^m\right)$$

This Markov chain is irreductible and has a unique stationary distribution p(x), which ensures the convergence of the Gibbs sampler to p(x).

sampling markov random fields

$$U(x) = -\beta_1 \sum_{\boxed{i\,|\,j}} x_i x_j - \beta_2 \sum_{\substack{\boxed{i} \\ \boxed{j}}} x_i x_j .$$

Some typical patterns drawn from this version are shown in Figure 3 for different combinations of β_1 and β_2. □

FIGURE 2. Samples from an Ising model on a lattice with first-order neighborhood system and increasing $\beta = 0$ (in that case X_is are i.i.d., and sampling is direct), 0.7, 0.9, 1.1, 1.5, and 2.

FIGURE 3. Samples from an anisotropic Ising model on a lattice with first-order neighborhood system and $(\beta_1, \beta_2) = (5, 0.5)$, $(5, 0.1)$, $(1, -1)$, $(-1, -1)$, respectively.

From Perez, 1998

MRF & inverse problems

Bayesian model:

- **Y: observed image**
- **X: image to be reconstructed**

$$p\left(\mathbf{x}|\mathbf{y}\right) \propto p\left(\mathbf{y}|\mathbf{x}\right) p\left(\mathbf{x}\right)$$

posterior — **observation likelihood** — **prior**

Different criterions:

- **Sampling or estimating the posterior p(x|y) (e.g., Gibbs sampler)**
- **MAP:** $\hat{x} = \arg\max_{x} p\left(\mathbf{x}|\mathbf{y}\right)$
- **MPM (Maximum Posterior Mode):** $\hat{x}_i = \arg\max_{x_i} p\left(\mathbf{x}_i|\mathbf{y}\right)$

From Perez, 1998

MRF & inverse problems

Bayesian model:

- **Y: observed image**
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$$p\left(\mathbf{x}|\mathbf{y}\right) \propto p\left(\mathbf{y}|\mathbf{x}\right) p\left(\mathbf{x}\right)$$

observation likelihood

$$\prod_i p\left(\mathbf{x}_i|\mathbf{y}_i\right)$$

MRF prior

$$\frac{1}{Z} \exp\left\{\sum_{c \in C} V_c(\mathbf{x}_c)\right\}$$

$$\Rightarrow \quad p\left(\mathbf{x}|\mathbf{y}\right) = \exp\left\{\sum_i \log p\left(\mathbf{y}_i|\mathbf{x}_i\right) + \sum_{c \in C} V_c(\mathbf{x}_c)\right\}$$

x|y is also a MRF

From Perez, 1998

MRF & inverse problems

Algorithm to solve for the MAP with MRF:

- **ICM (Iterative conditional mode): deterministic algorithm**
- **Gibbs sampler for x|y**
- **Graph-based algorithm (cf. Graph cut)**

From Perez, 1998

Additional resources

Markov Random fields and Images, P. Perez, 1998
http://www.cs.ubc.ca/~murphyk/Teaching/CS532c_Fall04/Papers/perez_cwi_quarterly.pdf

Video on Markov Random Fields on undirected graphs
https://www.youtube.com/watch?v=iBQkZdPHlCs